Choosing best shortcuts for a path

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Motivation

- We want to improve the network, but are constrained (budget)
- Our goal is to improve DAGs like Contraction Hierarchies

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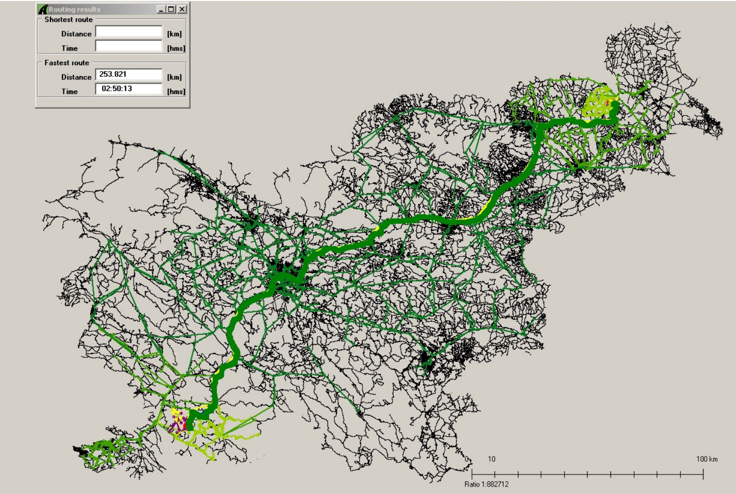
Motivation - example



Related problems

1. The Shortcut Problem: asks what is the best set of k additional edges (shortcuts) so that shortest paths will be preserved and the average hop length of paths will be minimal;
2. Multi-Constrained Shortest Path: find a path from s to t with lowest cost, subject to constraints;
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General model

Definition

Weighted multiobjective network design problem

$\mathbf{B} \in \mathbb{R}_0^{+d2}$	a vector of constraints ("budget")
$G = (V, E)$	a directed (multi)graph
$\mathbf{c} : E \mapsto \mathbb{R}_0^{+d1}$	costs
$\mathbf{b} : E \mapsto \mathbb{R}_0^{+d2}$	"building" costs
$\mathbf{w} : V \times V \mapsto \mathbb{R}_0^{+d1}$	weights

Determine $E_0 \subset E$ such that it satisfies the constrains $\mathbf{b}(E_0) \leq \mathbf{B}$ and achieves

$$\min_{E^S \subset E} \Big(\sum_{i,j} \mathbf{w}_{ij} \mathbf{c}(P_{G(V,E^S)}(i,j))\Big)$$

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Consider a path connecting some cities. We have a budget which allows construction of a few new road sections.

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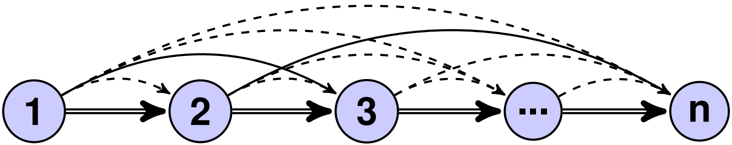

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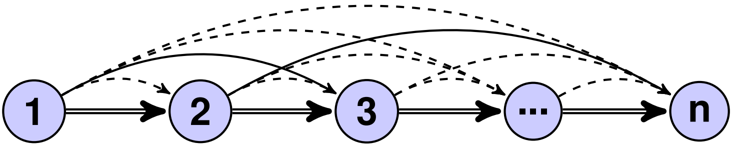

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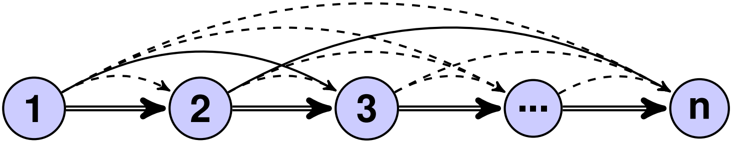

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Variants of the problem

Cost of using a shortcut ($b(e_s)$):

- $b(e_s) = 1$ for all allowed shortcuts,
- $b(e_s)$ can be different.

Weights (w_{ij}):

- $w_{1n} = 1$, other $w_{ij} = 0 \Rightarrow$ constrained shortest path;
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All subpaths distances

Subpath travelling costs

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$b = 1$ Constrained shortest path

$$c(1, n, B) = \min_{i,k}(c(1, i, k) + c(i, n, B - k))$$

$k = \lfloor B \rfloor$

Algorithm 1 Dynamic Programming CSP

```
for l = 0 to k do
    for i = 1 to n do
        for j = i to n do
            c(i, j, l) = min_{v,s}(c(i, v, s) + c(v, j, l - s))
            c(i, j, l) = min(c(i, j, l), c_{ij}^s)
        end for
    end for
end for
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Complexity is $O(k^2 n^3)$

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$b > 0$ Constrained shortest path

Even the simplest form (each arc replaced by a shortcut) is NP-complete, can be translated to knapsack problem

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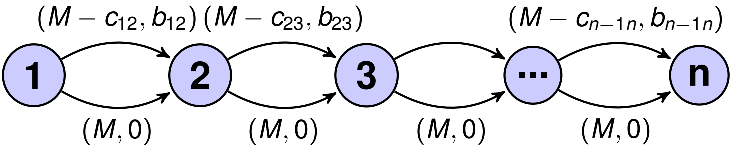

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Observations

The set of used shortcuts uniquely defines the resulting shortest path.

Two shortcuts are *compatible*, if (interiors of) the intervals of the indexes of the end vertices are not overlapping. The shortest path can only contain compatible edges.

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Lemma

If the sets are compatible, then
$g(S_O) \geq g(S_S) \Rightarrow c(P_{S_o}) \leq c(P_{S_s})$.

$\overline{P_{S_s}}$ are edges from P which are skipped in P_{S_s}.
$c(P_{S_s}) = c(P) + c(S_S) - c(\overline{P_{S_s}}) = c(P) - g(S_S)$
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$b > 0$ CSP - Greedy approach

Algorithm 4 Greedy CSP

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Assign each shortcut e_S its normalized gain g_N(e_s)
Sort shortcuts S by their normalized gain (descending)
S_O = {}, i = 1
while b(S_O) < B do
    if S(i) compatible with S_O then
        S_O = S_O ∪ S(i)
    end if
    i++
end while
for j = 1 to m do
    if g(S(j)) > g(S_O) then
        S_O = S(j)
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$2\frac{B}{b_{max}}$-approximation

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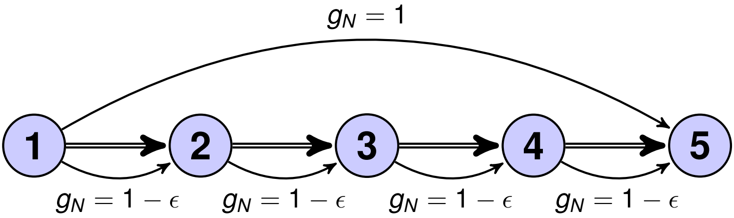

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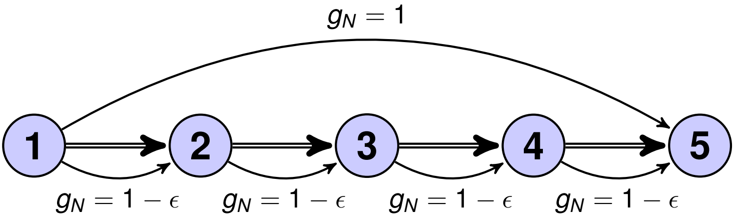

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- Special cases of CSP were discussed

Further work - find effective solutions for wider classes of graphs

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Composition of problems

CSP is composed of 2 problems:

- find shortest path
- find path, cheaper than B

Both are in P, but the composed problem is NP-hard

Open problem

What can we tell about the complexity of the composed problem, based on complexity of the subproblems and the type of composition?

Questions

Questions, comments, ideas, $\cdots$?